

RENORMALISED CONICAL ZETA VALUES

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ABSTRACT. Conical zeta values associated with rational convex polyhedral cones generalise multiple zeta values. We renormalise conical zeta values at poles by means of a generalisation of Connes and Kreimer's Algebraic Birkhoff Factorisation. This paper serves as a motivation for and an application of this generalised renormalisation scheme. The latter also yields an Euler-Maclaurin formula on rational convex polyhedral lattice cones which relates exponential sums to exponential integrals. When restricted to Chen cones, it reduces to Connes and Kreimer's Algebraic Birkhoff Factorisation for maps with values in the algebra of ordinary meromorphic functions in one variable.

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1. INTRODUCTION

Convergent conical zeta values

$$\zeta(C; \vec{s}) := \sum_{(n_1, \dots, n_k) \in C \cap \mathbb{Z}^k} n_1^{-s_1} \cdots n_k^{-s_k},$$

associated with a rational convex polyhedral cone $C \subset \mathbb{R}^k$ and $\vec{s} = (s_1, \dots, s_k) \in \mathbb{Z}^k$, which generalise multiple zeta values, were studied in [2]. The purpose of the present paper is to study their pole structure and to evaluate them at the poles.

A natural idea is to apply Connes and Kreimer's Algebraic Birkhoff Factorisation [1], see also [8]. One of the main ingredients needed for such a factorisation is a coalgebra structure on the source space - here the space of lattice cones - of the maps to be renormalised. In [4] we showed that the space of lattice cones carries a cogenerated, coaugmented, connected coalgebra structure; in the present paper, we show that this coalgebra can be enlarged to a differential coalgebra structure (Theorem 3.5).

Due to the geometric nature of convex cones, which is reflected in the specific coproduct built on the corresponding space of lattice cones, one cannot implement an univariate regularisation,

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namely one depending on a single parameter ε , as Connes and Kreimer did in their Algebraic Birkhoff Factorisation on Feynman graphs. The coproduct we use involves transverse cones built by means of an orthogonal projection, so we need a regularisation procedure which can be implemented for all cones under consideration, as well as their faces, together with the transverse cones to their faces. For a small enough family of lattice cones, such as the family of lattice Chen cones, their faces and the transverse lattice cones to their faces, one can use a univariate regularisation, in which case the regularised maps take values in Laurent series. One can then apply Connes and Kreimer's Algebraic Birkhoff Factorisation to the coalgebra of lattice cones modulo a minor adjustment due to the absence of a product on the space of such cones. However, to deal with general convex cones and the transverse cones to their faces, we need (Remark 4.1) a multivariate regularisation (Eq. (21)) which involves a vector parameter $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_k) \in \mathbb{C}^k$. The regularised maps we build this way take values in the space of multivariate meromorphic germs at zero with linear poles (Proposition-Definition 5.1), which we investigated in [2].

More precisely, to renormalise conical zeta values associated lattice cones (C, Λ) at their poles, we implement a generalisation (Theorem 2.5) of Connes and Kreimer's Algebraic Birkhoff Factorisation device [1] to the map on the coalgebra of lattice cones defined by an exponential sum $S((C, \Lambda))$ on the lattice cones (C, Λ) . The generalisation is two fold:

- the exponential sums we want to factorise act on the colagebra of lattice cones, which is only equipped with a partial product, so the source space is not any longer a Hopf algebra.
- the exponential sums have values in the algebra of multivariate meromorphic functions, so the target space is not any longer a Rota-Baxter algebra.

This was carried out in [4]. In the present paper, we further generalise the coalgebra of cones, and consider the Algebraic Birkhoff Factorisation with additional differential structures. Indeed, in view of renormalising conical zeta values, not only do we need to renormalise the exponential sums but also their derivatives with respect to the regularisation parameter. Hence the need for an additional differential structure which comes with a decoration \vec{s} leading to coloured lattice cones (C, Λ, \vec{s}) ¹.

This renormalisation procedure (Theorem 5.6) implemented on the exponential sums $S((C, \Lambda); \vec{s})$ associated with coloured lattice cones $(C, \Lambda; \vec{s})$ implies an Euler-Maclaurin formula (Eqn. 28) on lattice cones [4] which relates exponential sums to the corresponding exponential integrals. The renormalised conical zeta values $\zeta^{\text{ren}}((C, \Lambda), \vec{s})$ associated with a coloured lattice cone $((C, \Lambda); \vec{s})$ are derived (Eqn. (20)) from the factors entering the factorisation formula of the associated exponential sum $S((C, \Lambda); \vec{s})$.

On the smaller coalgebra of lattice Chen cones, the multivariate regularisation procedure implemented on the algebra of all convex lattice cones, can be reduced to a univariate regularisation procedure by specifying one direction of regularisation $\vec{\varepsilon} := \vec{d}\varepsilon$ for some fixed vector \vec{d} . We show (Proposition 6.2) how in the case of lattice Chen cones, specialising to an univariate regularisation procedure in specifying a direction \vec{d} , our renormalisation procedure amounts to the usual Algebraic Birkhoff Factorisation on the maps given by the exponential sums on the lattice cones, with values in Laurent series, thus independent of the choice of the direction \vec{d} . As a by-product, our geometric renormalisation procedure therefore yields renormalised multiple zeta values at negative integers obtained as renormalised conical zeta values associated with lattice Chen cones. However, these renormalised multiple zeta values do not satisfy the stuffle relations [5] due to the use of the coproduct on Chen cones which involves an orthogonal complement

¹Note the difference with decorated lattices cones in [2]

map. Thus, the renormalised multiple zeta values we obtain here by a geometric approach as particular instances of conical zeta values, differ from the ones derived in [9] and [6] by an alternative algebro-combinatorial approach. As observed in [4], the renormalised conical values derived here by means of a multivariate Algebraic Birkhoff Factorisation, can alternatively be derived directly from the derivatives of the exponential sums on cones by means of the projection onto the holomorphic part of the meromorphic germs they give rise to. In this respect, the multivariate parametrisation approach-imposed here by the geometric nature of the cones- bares over the univariate one, the advantage that renormalisation then amounts to a projection on the target space of multivariate meromorphic germs without the need for an Algebraic Birkhoff Factorisation. So, not only is the multivariate approach necessary when dealing with the space of all cones, but it is also very useful in so far as it provides a way to circumvent the use of an Algebraic Birkhoff Factorisation all together.

2. GENERALISED ALGEBRAIC BIRKHOFF FACTORISATION

Let us first recall the Algebraic Birkhoff Factorisation of Connes and Kreimer's renormalisation scheme [1], which we shall then generalise in order to later renormalise conical zeta values at poles.

Theorem 2.1. *Let H be a commutative connected graded Hopf algebra and (R, P) be a Rota-Baxter algebra of weight -1 , $\phi : H \rightarrow R$ be an algebra homomorphism.*

- (a) *There are algebra homomorphisms $\phi_- : H \rightarrow \mathbf{k} + P(R)$ and $\phi_+ : H \rightarrow \mathbf{k} + (1 - P)(R)$ such that*

$$\phi = \phi_-^{*(-1)} * \phi_+.$$

Here $\phi_-^{(-1)}$ is the inverse of ϕ_- with respect to the convolution product.*

- (b) *If $P^2 = P$, then the decomposition in (a) is unique.*

On the one hand, in [4], we generalised the Algebraic Birkhoff Factorisation of Connes-Kreimer's renormalisation scheme for connected coalgebras without the need for either a Hopf algebra in the source or a Rota-Baxter algebra in the target. On the other hand, we provided the following differential variant in [7].

Theorem 2.2. *If (H, d) is in addition a differential Hopf algebra, (R, P, ∂) is a commutative differential Rota-Baxter algebra, and ϕ is a differential algebra homomorphism, then ϕ_- and ϕ_+ are also differential algebra homomorphisms.*

In order to explore the structure of renormalised conical zeta values, we combine these two generalisations.

Definition 2.3. A **differential cograded, coaugmented, connected coalgebra** is a cograded, coaugmented, connected coalgebra $\left(\mathbf{C} = \bigoplus_{n \geq 0} \mathbf{C}^{(n)}, \Delta, \varepsilon, u\right)$ with linear maps $\delta_\sigma : \mathbf{C} \rightarrow \mathbf{C}$ for σ in an index set Σ such that

$$(1) \quad \Delta \delta_\sigma = (\text{id} \otimes \delta_\sigma + \delta_\sigma \otimes \text{id}) \Delta, \quad \delta_\sigma(\mathbf{C}^{(n)}) \subseteq \mathbf{C}^{(n+1)}, \quad \delta_\sigma \delta_\tau = \delta_\tau \delta_\sigma, \quad \sigma, \tau \in \Sigma.$$

The linear maps $\delta_\sigma, \sigma \in \Sigma$, are called **coderivations** on \mathbf{C} .

It follows from the definition that δ_σ stabilises $\ker \varepsilon$. Recall the counit property of ε for Δ :

$$(2) \quad \beta_\ell = (\varepsilon \otimes \text{id})\Delta, \quad \beta_r = (\text{id} \otimes \varepsilon)\Delta,$$

where

$$\beta_\ell : \mathbf{C} \rightarrow \mathbf{k} \otimes \mathbf{C}, x \mapsto 1 \otimes x, \quad \beta_r : \mathbf{C} \rightarrow \mathbf{C} \otimes \mathbf{k}, x \mapsto x \otimes 1,$$

with

$$\beta_\ell^{-1} : \mathbf{k} \otimes \mathbf{C} \rightarrow \mathbf{C}, a \otimes x \mapsto ax, \quad \beta_r^{-1} : \mathbf{C} \otimes \mathbf{k} \rightarrow \mathbf{C}, x \otimes a \mapsto ax.$$

Lemma 2.4. *For a differential cogenerated, coaugmented, connnected coalgebra $(C, \Delta, \varepsilon, u)$ with coderivations $\delta_\sigma, \sigma \in \Sigma$, we have $\varepsilon \delta_\sigma = 0$.*

Proof. Apply $\varepsilon \otimes \varepsilon$ to the two sides of the identity $\Delta \delta_\sigma = (\text{id} \otimes \delta_\sigma + \delta_\sigma \otimes \text{id}) \Delta$. By the counit property in Eq. (2), on the left hand side we have

$$(\varepsilon \otimes \varepsilon) \Delta \delta_\sigma = (\varepsilon \otimes \text{id})(\text{id} \otimes \varepsilon) \Delta \delta_\sigma = (\varepsilon \otimes \text{id}) \beta_r \delta_\sigma = (\varepsilon \delta_\sigma \otimes \text{id}) \beta_r.$$

Similarly on the right hand side we have

$$(\varepsilon \otimes \varepsilon)(\text{id} \otimes \delta_\sigma + \delta_\sigma \otimes \text{id}) \Delta = (\varepsilon \otimes \varepsilon \delta_\sigma) \Delta + (\varepsilon \delta_\sigma \otimes \varepsilon) \Delta = (1 \otimes \varepsilon \delta_\sigma) \beta_\ell + (\varepsilon \delta_\sigma \otimes 1) \beta_r.$$

Thus we obtain $(1 \otimes \varepsilon \delta_\sigma) \beta_\ell = 0$. Hence $\varepsilon \delta_\sigma = 0$. \square

As we shall argue later on, the renormalisation of conical zeta values requires the following generalised version of this theorem [4] and its differential variant, to connected coalgebras in the source space, which are not necessarily Hopf algebras and algebras in the target space which are not necessarily Rota-Baxter algebras.

Theorem 2.5. *Let $\mathbf{C} = \bigoplus_{n \geq 0} \mathbf{C}^{(n)}$ be a differential cogenerated, coaugmented, connnected coalgebra with coderivations $\delta_\sigma, \sigma \in \Sigma$. Let A be a unitary differential algebra with derivations $\partial_\sigma, \sigma \in \Sigma$. Let $A = A_1 \oplus A_2$ be a linear decomposition such that $1_A \in A_1$ and*

$$\partial_\sigma(A_i) \subseteq A_i, \quad i = 1, 2, \quad \sigma \in \Sigma.$$

Let P be the projection of A to A_1 along A_2 . Given $\phi \in \mathcal{G}(\mathbf{C}, A)$ such that $\partial_\sigma \phi = \phi \delta_\sigma, \sigma \in \Sigma$, define maps $\varphi_i \in \mathcal{G}(\mathbf{C}, A), i = 1, 2$, by the following recursive formulae on $\ker \varepsilon$:

$$(3) \quad \varphi_1(x) = -P\left(\varphi(x) + \sum_{(x)} \varphi_1(x') \varphi(x'')\right),$$

$$(4) \quad \varphi_2(x) = (\text{id}_A - P)\left(\varphi(x) + \sum_{(x)} \varphi_1(x') \varphi(x'')\right).$$

(a) *We have $\varphi_i(\ker \varepsilon) \subseteq A_i$ (hence $\varphi_i : \mathbf{C} \rightarrow \mathbf{k}1_A + A_i$) and $\delta_\sigma \varphi_i = \varphi_i \delta_\sigma, i = 1, 2, \sigma \in \Sigma$. Moreover,*

$$(5) \quad \varphi = \varphi_1^{*(-1)} * \varphi_2$$

(b) *φ_1 and φ_2 are the unique maps in $\mathcal{G}(\mathbf{C}, A)$ such that $\varphi_i(\ker \varepsilon) \subseteq A_i$ for $i = 1, 2$, and Eq. (5) holds.*

(c) *If moreover A_1 is a subalgebra of A then $\varphi_1^{*(-1)}$ lies in $\mathcal{G}(\mathbf{C}, A_1)$.*

Remark 2.6. When the coderivations δ_σ and derivations $\partial_\sigma, \sigma \in \Sigma$, are taken to be the zero maps, we obtain a generalisation of the Algebraic Birkhoff Factorisation of Connes and Kreimer [1] which does not involve the differential structure, for maps from a connected coalgebra (which is not necessarily equipped with a product) to a decomposable unitary algebra (which does not necessarily decompose into a sum of two subalgebras). This also generalises the differential Algebraic Birkhoff Factorisation in [7].

Proof. (a) The inclusion $\varphi_i(\ker \varepsilon) \subseteq A_i, i = 1, 2$, follows from the definitions. Further

$$\varphi_2(x) = (\text{id}_A - P)\left(\varphi(x) + \sum_{(x)} \varphi_1(x')\varphi(x'')\right) = \varphi(x) + \varphi_1(x) + \sum_{(x)} \varphi_1(x')\varphi(x'') = (\varphi_1 * \varphi)(x).$$

Since $\varphi_1(J) = 1_A$, φ_1 is invertible for the convolution product in A as a result of [7, Theorem 3.2] applied to φ_1 , from which Eq. (5) then follows.

To verify $\partial_\sigma \varphi_i = \varphi_i \delta_\sigma, i = 1, 2, \sigma \in \Sigma$, we first establish $P\partial_\sigma = \partial_\sigma P$ by verifying it on A_1 and A_2 . We then implements the same inductive argument as in [7, Theorem 3.2].

(b) Suppose there are $\psi_i \in \mathcal{G}(\mathbf{C}, A), i = 1, 2$, with $\psi_i(\ker \varepsilon) \subseteq A_i$ such that $\varphi = \psi_1^{*(-1)} * \psi_2$. We prove $\varphi_i(x) = \psi_i(x)$ for $i = 1, 2, x \in \mathbf{C}^{(k)}$ by induction on $k \geq 0$. These equations hold for $k = 0$. Assume that the equations hold for $x \in \mathbf{C}^{(k)}$ where $k \geq 0$. For $x \in \mathbf{C}^{(k+1)} \subseteq \ker(\varepsilon)$, by $\varphi_2 = \varphi_1 * \varphi$ and $\psi_2 = \psi_1 * \varphi$, we have

$$\varphi_2(x) = \varphi_1(x) + \varphi(x) + \sum_{(x)} \varphi_1(x')\varphi(x'')$$

and similarly for ψ , namely,

$$\psi_2(x) = \psi_1(x) + \varphi(x) + \sum_{(x)} \psi_1(x')\varphi(x''),$$

where we have made use of $\varphi_1(J) = \psi_1(J) = \varphi(J) = 1_A$. Hence by the induction hypothesis, we have

$$\varphi_2(x) - \psi_2(x) = \varphi_1(x) - \psi_1(x) + \sum_{(x)} (\varphi_1(x') - \psi_1(x'))\varphi(x'') = \varphi_1(x) - \psi_1(x) \in A_1 \cap A_2 = \{0\}.$$

Thus $\varphi_i(x) = \psi_i(x)$ for all $x \in \ker(\varepsilon), i = 1, 2$.

(c) If A_1 is a subalgebra, then it follows from [8, Proposition II.3.1] applied to A_1 that φ_1 is invertible in A_1 . \square

3. A DIFFERENTIAL COALGEBRAIC STRUCTURE ON LATTICE CONES

We now apply the general setup in the last section to lattice cones.

3.1. Lattice cones. We begin with recalling the notion and basic properties of lattice cones. See [4] for details. In a finite dimensional real vector space, a **lattice** is a finitely generated subgroup which spans the whole space over \mathbb{R} . Such a pair, namely a real vector space equipped with a lattice is called a **lattice vector space**. Let $V_1 \subset V_2 \subset \dots$ be a family of finite dimensional real vector spaces, and let Λ_k be a lattice in V_k such that $\Lambda_k = \Lambda_{k+1} \cap V_k$. The vector space $V := \bigcup_{k=1}^{\infty} V_k$ and the corresponding lattice $\Lambda := \bigcup_{k=1}^{\infty} \Lambda_k$ are equipped with their natural filtration. Such a pair (V, Λ) is called a **filtered lattice space**. Usually we work in $(\mathbb{R}^\infty, \mathbb{Z}^\infty)$ with $V_k = \mathbb{R}^k$, Λ_k the standard lattice \mathbb{Z}^k , and $\{e_1, e_2, \dots\}$ the standard basis.

For a filtered lattice space (V, Λ) , a point/vector in Λ is called an **lattice point/vector**, a rational multiple of an integer point/vector is called a **rational lattice point/vector**.

For a subset S of V , let $\text{lin}(S)$ denote its \mathbb{R} -linear span. In this paper, we only consider subspaces of V spanned by rational lattice vectors.

Let $V := \bigcup_{k \geq 1} V_k$ with $\Lambda = \bigcup_{k \geq 1} \Lambda_k$ be a filtered lattice space. An **inner product** $Q(\cdot, \cdot) = (\cdot, \cdot)$ on V is a sequence of inner products

$$Q_k(\cdot, \cdot) = (\cdot, \cdot)_k : V_k \otimes V_k \rightarrow \mathbb{R}, \quad k \geq 1,$$

that is compatible with the inclusion $j_k : V_k \hookrightarrow V_{k+1}$ and whose restriction to $\Lambda \otimes \mathbb{Q}$ and hence Λ takes values in \mathbb{Q} . A filtered lattice space together with an inner product is called a **filtered lattice Euclidean space**.

Let L be a subspace of V_k . Set

$$L^{\perp_k^Q} := \{v \in V_k \mid Q_k(v, u) = 0 \text{ for all } u \in L\}.$$

The inner product Q_k gives the direct sum decomposition $V_k = L \oplus L^{\perp_k^Q}$ and hence the orthogonal projection

$$(6) \quad \pi_{k, L^{\perp}}^Q : V_k \rightarrow L^{\perp_k^Q}$$

along L as well as an isomorphism

$$\iota_{k, L}^Q : V_k/L \rightarrow L^{\perp_k^Q}.$$

Also, the induced isomorphism $Q_k^* : V_k \rightarrow V_k^*$ yields an embedding $V_k^* \hookrightarrow V_{k+1}^*$. We refer to the direct limit $V^{\otimes} := \bigcup_{k=0}^{\infty} V_k^* = \varinjlim V_k^*$ as the **filtered dual space** of V . We will fix an inner product $Q(\cdot, \cdot) = (\cdot, \cdot)$ and drop the superscript Q to simplify notations.

We collect basic definitions and facts on lattice cones that will be used in this paper, see [2] for a detailed discussion.

- (a) By a **cone** in V_k we mean a **closed convex (polyhedral) cone** in V_k , namely the convex set

$$(7) \quad \langle v_1, \dots, v_n \rangle := \mathbb{R}\{v_1, \dots, v_n\} = \mathbb{R}_{\geq 0}v_1 + \dots + \mathbb{R}_{\geq 0}v_n,$$

where $v_i \in V_k, i = 1, \dots, n$.

- (b) A cone is called **rational** if the v_i 's in Eq. (7) are in Λ_k . This is equivalent to requiring that the vectors are in $\Lambda_k \otimes \mathbb{Q}$.
- (c) A Chen cone is any smooth cone in \mathbb{R}^{∞} of the form $\langle e_1, e_1 + e_2, \dots, e_1 + \dots + e_k \rangle$ and is denoted by C_k^{Chen} . Note that the faces of a Chen cone $\langle e_1, e_1 + e_2, \dots, e_1 + \dots + e_k \rangle$ are of the form $\langle e_1 + \dots + e_{i_1}, e_1 + \dots + e_{i_2}, \dots, e_1 + \dots + e_{i_l} \rangle$ for some indices $1 \leq i_1 < \dots < i_l \leq k$, so they are not Chen cones.
- (d) A **subdivision** of a cone C is a set $\underline{C} = \{C_1, \dots, C_r\}$ of cones such that

$$(i) \quad C = \bigcup_{i=1}^r C_i,$$

(ii) C_1, \dots, C_r have the same dimension as C , and

(iii) C_1, \dots, C_r intersect along their faces, i.e., $C_i \cap C_j$ is a face of both C_i and C_j .

We will use $\mathcal{F}^o(\underline{C})$ denote the set of faces of C_1, \dots, C_r that are not contained in any proper face of C .

- (e) A **lattice cone** in V_k is a pair (C, Λ_C) with C a cone in V_k and Λ_C a lattice in $\text{lin}(C)$ generated by rational vectors.
- (f) A **face** of a lattice cone (C, Λ_C) is the lattice cone (F, Λ_F) where F is a face of C and $\Lambda_F := \Lambda_C \cap \text{lin}(F)$.
- (g) A **primary generating set** of a lattice cone (C, Λ_C) is a generating set $\{v_1, \dots, v_n\}$ of C such that
- (i) $v_i \in \Lambda_C, i = 1, \dots, n$,
- (ii) there is no real number $r_i \in (0, 1)$ such that $r_i v_i$ lies in Λ_C , and
- (iii) none of the generating vectors v_i is a positive linear combination of the others.

- (h) A lattice cone (C, Λ_C) is called **strongly convex** (resp. **simplicial**) if C is. A lattice cone (C, Λ_C) is called **smooth** if the additive monoid $\Lambda_C \cap C$ has a monoid basis. In

other words, there are linearly independent rational lattice vectors v_1, \dots, v_ℓ such that $\Lambda_C \cap C = \mathbb{Z}_{\geq 0}\{v_1, \dots, v_\ell\}$.

- (i) A **subdivision** of a lattice cone (C, Λ_C) is a set of lattice cones $\{(C_i, \Lambda_{C_i}) \mid 1 \leq i \leq r\}$ such that $\{C_i \mid 1 \leq i \leq r\}$ is a subdivision of C and $\Lambda_{C_i} = \Lambda_C$ for all $1 \leq i \leq r$.
- (j) Let F be a face of a cone $C \subseteq V_k$. The **transverse cone** $t(C, F)$ to F is the projection $\pi_{k, F^\perp}(C)$ of C in $\text{lin}(F)^\perp \subseteq V_k$, where $\pi_{k, F^\perp} = \pi_{k, \text{lin}(F)^\perp}$.
- (k) Let (F, Λ_F) be a face of the lattice cone (C, Λ_C) . The **transverse lattice cone** $(t(C, F), \Lambda_{t(C, F)})$ along the face (F, Λ_F) is the projection of (C, Λ_C) on $\text{lin}(F)^\perp \subseteq V_k$. More precisely, let $\pi_{F^\perp} : V_k \rightarrow \text{lin}(F)^\perp$ be the projection, then

$$(8) \quad (t(C, F), \Lambda_{t(C, F)}) := (\pi_{F^\perp}(C), \pi_{F^\perp}(\Lambda_C)).$$

We also use the notation $t((C, \Lambda_C), (F, \Lambda_F))$ to denote the transverse lattice cone.

As in the case of ordinary cones, we have the following property.

Proposition 3.1. *Any lattice cone can be subdivided into smooth lattice cones.*

Proof. For a given lattice cone (D, Λ_C) in a simplicial subdivision of a lattice cone (C, Λ_C) with its primary generating set $\{v_1, \dots, v_n\}$, we write $v_i = \sum_{j=1}^n a_{ij}u_j$, $a_{ij} \in \mathbb{Z}$, $i = 1, \dots, n$, where $\{u_1, \dots, u_n\}$ is a basis of Λ_C . The absolute value of the determinant $w_D = |v_1, \dots, v_n| := |\det(a_{ij})|$ lies in $\mathbb{Z}_{\geq 1}$ and is independent of the choice of a basis $\{u_1, \dots, u_n\}$ of Λ_C . Further w_D is equal to one if and only if (D, Λ_C) is smooth.

We now prove the proposition by contradiction. Suppose (C, Λ_C) is a lattice cones that cannot be subdivided into smooth lattice cones. Then for any simplicial subdivision $\underline{C} := \{(C_i, \Lambda_{C_i})\}$ of (C, Λ_C) , we have

$$w_{\underline{C}} := \max\{w_{C_i}\} > 1 \quad \text{and} \quad n_{\underline{C}} := \max\{|i|, w_{C_i} = w_{\underline{C}}\} \geq 1.$$

Choose a simplicial subdivision \underline{C} of (C, Λ_C) with $w_{\underline{C}}$ minimal and then among those, one with $n_{\underline{C}}$ minimal. We will construct a subdivision of (C, Λ_C) that refines \underline{C} . Let $D = \langle v_1, \dots, v_n \rangle$ be a cone in \underline{C} with $w_D = w_{\underline{C}}$. Since $w_D > 1$, the lattice cone (D, Λ_C) is not smooth. So $\{v_1, \dots, v_n\}$ is not a lattice basis of $\Lambda_C \cap D$. Note that the set $\{v_1, \dots, v_n\} \cup \left(\left(\sum_{i=1}^n [0, 1)v_i \right) \cap \Lambda_C \right)$ spans $\Lambda_C \cap D$ as a monoid. So there is a vector $0 \neq v_D = \sum_{i=1}^n c_i v_i \in \Lambda_C$ with $c_i \in [0, 1)$ rational.

Reordering v_i , we can assume that $c_i \neq 0$ for $i = 1, \dots, k$, and $c_i = 0$ for $i = k+1, \dots, n$. We now use the vector $v_D = \sum_{i=1}^k c_i v_i$ to subdivide the cones. Let $C_i = \langle v_1, \dots, v_k, v_{k+1}^i, \dots, v_n^i \rangle$, $i = 1, \dots, s$, be all the cones arising in the subdivision \underline{C} that contain $\langle v_1, \dots, v_k \rangle$ as a face, with $C_1 = D$. Then the set of cones

$$\{C_i, i > s\} \cup \{C_{ij} := \langle v_1, \dots, \check{v}_j^D, \dots, v_k, v_{k+1}^i, \dots, v_n^i \rangle \mid j = 1, \dots, k, i = 1, \dots, s\},$$

where \check{v}_j^D means v_j has been replaced by v_D , yields a new subdivision \underline{C}' of C .

For elements in \underline{C}' , the numbers w_{C_i} , $i > s$ coincide. For $i = 1, \dots, s$ and $j = 1, \dots, k$,

$$|v_1, \dots, \check{v}_j^D, \dots, v_k, v_{k+1}^i, \dots, v_n^i| = c_j |v_1, \dots, v_k, v_{k+1}^i, \dots, v_n^i| < |v_1, \dots, v_k, v_{k+1}^i, \dots, v_n^i| = w_{C_i}.$$

So $w_{C_{ij}} < w_{\underline{C}}$. Therefore either $w_{\underline{C}'} < w_{\underline{C}}$, or $w_{\underline{C}'} = w_{\underline{C}}$ and $n_{\underline{C}'} < n_{\underline{C}}$. This gives the desired contradiction. \square

Proposition 3.2. [4] *Transverse cones enjoy the following properties. Let F be a face of a cone C .*

- (a) **(Transitivity)** $t(C, F) = t(t(C, F'), t(F, F'))$ if F' is a face of F .
- (b) **(Compatibility with the partial order)** We have $\{H \leq t(C, F)\} = \{t(G, F) \mid F \leq G \leq C\}$.
- (c) **(Compatibility with the dimension filtration)** $\dim(C) = \dim(F) + \dim(t(C, F))$ for any face F of C .

To the first two properties correspond similar properties for lattice cones.

- (d) **(Transitivity)** $t((C, \Lambda_C), (F, \Lambda_F)) = t(t((C, \Lambda_C), (F', \Lambda_{F'})), t((F, \Lambda_F), (F', \Lambda_{F'})))$ if $(F', \Lambda_{F'})$ is a face of (F, Λ_F) .
- (e) **(Compatibility with the partial order)** We have

$$\{(H, \Lambda_H) \leq t((C, \Lambda_C), (F, \Lambda_F))\} = \{t((G, \Lambda_G), (F, \Lambda_F)) \mid (F, \Lambda_F) \leq (G, \Lambda_G) \leq (C, \Lambda_C)\}.$$

3.2. The coalgebra of lattice cones. Let \mathfrak{C}_k denote the set of lattice cones in V_k , $k \geq 1$. The natural inclusions $\mathfrak{C}_k \rightarrow \mathfrak{C}_{k+1}$ induced by the natural inclusions $V_k \rightarrow V_{k+1}$, $\Lambda_k \rightarrow \Lambda_{k+1}$, $k \geq 1$, give rise to the direct limit $\mathfrak{C} = \varinjlim \mathfrak{C}_k = \cup_{k \geq 1} \mathfrak{C}_k$.

We equip the \mathbb{Q} -linear space $\mathbb{Q}\mathfrak{C}$ generated by \mathfrak{C} with a coproduct by means of transverse lattice cones. The maps

$$(9) \quad \Delta : \mathbb{Q}\mathfrak{C} \longrightarrow \mathbb{Q}\mathfrak{C} \otimes \mathbb{Q}\mathfrak{C}, \quad (C, \Lambda_C) \mapsto \sum_{F \leq C} (t(C, F), \Lambda_{t(C, F)}) \otimes (F, \Lambda_C \cap \text{lin}(F)),$$

$$(10) \quad \varepsilon : \mathbb{Q}\mathfrak{C} \longrightarrow \mathbb{Q}, \quad (C, \Lambda_C) \mapsto \begin{cases} 1, & C = \{0\}, \\ 0, & C \neq \{0\}, \end{cases}$$

and

$$(11) \quad u : \mathbb{Q} \rightarrow \mathbb{Q}\mathfrak{C}, \quad 1 \mapsto (\{0\}, \{0\}).$$

define a cograded, coaugmented, connected coalgebra with the grading

$$(12) \quad \mathbb{Q}\mathfrak{C} = \bigoplus_{n \geq 0} \mathbb{Q}\mathfrak{C}^{(n)},$$

where

$$\mathfrak{C}^{(n)} := \{(C, \Lambda_C) \in \mathfrak{C} \mid \dim C = n\}, \quad n \geq 0.$$

Corollary 3.3. *For a given lattice cone (C, Λ_C) , the subspace*

$$\bigoplus_{F \leq C} \mathbb{Q}(F, \Lambda_F) \oplus \bigoplus_{F' \leq F \leq C} \mathbb{Q}(t(F, F'), \Lambda_{t(F, F')})$$

of $\mathbb{Q}\mathfrak{C}$ is a subcoalgebra of $\mathbb{Q}\mathfrak{C}$.

Now we work in $(\mathbb{R}^\infty, \mathbb{Z}^\infty)$ with $V_k = \mathbb{R}^k$, Λ_k the standard lattice \mathbb{Z}^k , and $\{e_1, e_2, \dots\}$ the standard basis. Let $\mathbb{Z}_{\leq 0}^\infty = \varinjlim \mathbb{Z}_{\leq 0}^k$. For any element $\vec{s} = (s_i) \in \mathbb{Z}_{\leq 0}^\infty$, we set $|\vec{s}| := \sum |s_i|$.

On the space $\mathbb{Q}\mathfrak{D}\mathfrak{C}$ freely generated by the set

$$\mathfrak{D}\mathfrak{C} := \mathfrak{C} \times \mathbb{Z}_{\leq 0}^\infty$$

of **coloured lattice cones**, there is a family of linear operators

$$(13) \quad \delta_i : \mathbb{Q}\mathfrak{D}\mathfrak{C} \rightarrow \mathbb{Q}\mathfrak{D}\mathfrak{C} \quad ((C, \Lambda_C); \vec{s}) \mapsto ((C, \Lambda_C); \vec{s} - e_i).$$

By an inductive argument on $|\vec{s}|$, we obtain

Lemma 3.4. For $(C, \Lambda_C) \in \mathfrak{C}$, $k \geq 1$ and $\vec{s} \in \mathbb{Z}_{\leq 0}^k$, we have

$$((C, \Lambda_C); \vec{s}) = \delta_1^{-s_1} \cdots \delta_k^{-s_k} ((C, \Lambda_C); \vec{0}).$$

We next extend the coproduct Δ on $\mathbb{Q}\mathfrak{C}$ to a coproduct on $\mathbb{Q}\mathfrak{D}\mathfrak{C}$, still denoted by Δ . We proceed by induction on $n := |\vec{s}|$. For $n = 0$, we have $\vec{s} = \vec{0}$ and define

$$\Delta((C, \Lambda_C); \vec{0}) = \sum \left((C_{(1)}, \Lambda_{C_{(1)}}), \vec{0} \right) \otimes \left((C_{(2)}, \Lambda_{C_{(2)}}), \vec{0} \right),$$

using the coproduct $\Delta(C, \Lambda_C) = \sum (C_{(1)}, \Lambda_{C_{(1)}}) \otimes (C_{(2)}, \Lambda_{C_{(2)}})$ on $\mathbb{Q}\mathfrak{C}$ define in Eq. (9).

Assume that the coproduct Δ has been defined for $((C, \Lambda_C); \vec{s})$ with $|\vec{s}| = \ell$ for $\ell \geq 0$. Consider $((C, \Lambda_C), \vec{s}) \in \mathfrak{D}\mathfrak{C}$ with $\vec{s} \in \mathbb{Z}_{\leq 0}^k$, $|\vec{s}| = \ell + 1$. Then there is some i such that $s_i \leq -1$ and we define

$$(14) \quad \Delta((C, \Lambda_C); \vec{s}) = (\Delta \delta_i)((C, \Lambda_C); \vec{s} + e_i) := (D_i \Delta)((C, \Lambda_C); \vec{s} + e_i),$$

where $D_i = \delta_i \otimes 1 + 1 \otimes \delta_i$. Explicitly, we have

$$(15) \quad \Delta((C, \Lambda_C); \vec{s}) = D_1^{-s_1} \cdots D_k^{-s_k} \Delta((C, \Lambda_C); \vec{0}).$$

The counit ε in Eq. (2) is trivially extended to a map on $\mathbb{Q}\mathfrak{D}\mathfrak{C}$ for which we use the same notation

$$(16) \quad \varepsilon : \mathbb{Q}\mathfrak{D}\mathfrak{C} \rightarrow \mathbb{Q}, \quad \varepsilon((C, \Lambda_C); \vec{s}) = \begin{cases} 1, & ((C, \Lambda_C); \vec{s}) = ((\{0\}, \{0\}); \vec{0}), \\ 0, & \text{otherwise.} \end{cases}$$

In particular, ε vanishes on cones of positive dimension. In view of the canonical embedding $\mathfrak{C} \rightarrow \mathfrak{D}\mathfrak{C}$, the unit u defined in Eq. (2) can be seen as the map

$$(17) \quad u : \mathbb{Q} \rightarrow \mathbb{Q}\mathfrak{D}\mathfrak{C}, \quad 1 \mapsto ((\{0\}, \{0\}); 0).$$

Denote

$$(18) \quad \mathfrak{D}\mathfrak{C}^{(n)} := \left\{ ((C, \Lambda_C); \vec{s}) \mid \dim C + |\vec{s}| = n \right\}, \quad n \geq 0.$$

Then by definition, we have $\mathfrak{D}\mathfrak{C}^{(0)} = \{((\{0\}, \{0\}); 0)\}$ and $\delta_i(\mathfrak{D}\mathfrak{C}^{(n)}) \subseteq \mathfrak{D}\mathfrak{C}^{(n+1)}$, $n \geq 0$.

Theorem 3.5. Let Δ, ε, u be as defined in Eqs. (15), (16) and (17). Equipped with the grading as in Eq. (18) and the derivations in Eq. (13), the quadruple $(\mathbb{Q}\mathfrak{D}\mathfrak{C}, \Delta, \varepsilon, u)$ becomes a differential cogenerated, coaugmented, connected coalgebra.

Proof. The first equation in Eq. (1) is just Eq. (14). The other equations follow from the definitions.

We prove the coassociativity by induction on $|\vec{s}|$ with the initial case $|\vec{s}| = 0$ given by the coassociativity of Δ on $\mathbb{Q}\mathfrak{C}$, where a lattice cone $(C, \Lambda_C) \in \mathfrak{C}$ is identified with $((C, \Lambda_C); \vec{0})$.

Suppose the coassociativity has been proved for vectors $\vec{s} \in \mathbb{Z}_{\leq 0}^k$ with $|\vec{s}| = n \geq 0$ and let $\vec{s} \in \mathbb{Z}_{\leq 0}^k$ with $|\vec{s}| = n + 1$. Then there is some index i with $s_i \leq -1$. By the induction hypothesis, we have $(\Delta \otimes \text{id})\Delta((C, \Lambda_C); \vec{s} + \vec{e}_i) = (\text{id} \otimes \Delta)\Delta((C, \Lambda_C); \vec{s} + \vec{e}_i)$. It follows that

$$\begin{aligned} (\Delta \otimes \text{id})\Delta((C, \Lambda_C); \vec{s}) &= (\Delta \otimes \text{id})D_i\Delta((C, \Lambda_C); \vec{s} + \vec{e}_i) \\ &= (\delta_i \otimes \text{id} \otimes \text{id} + \text{id} \otimes \delta_i \otimes \text{id} + \text{id} \otimes \text{id} \otimes \delta_i)(\Delta \otimes \text{id})\Delta((C, \Lambda_C); \vec{s} + \vec{e}_i) \\ &= (\delta_i \otimes \text{id} \otimes \text{id} + \text{id} \otimes \delta_i \otimes \text{id} + \text{id} \otimes \text{id} \otimes \delta_i)(\text{id} \otimes \Delta)\Delta((C, \Lambda_C); \vec{s} + \vec{e}_i) \\ &= (\text{id} \otimes \Delta)D_i\Delta((C, \Lambda_C); \vec{s} + \vec{e}_i) \\ &= (\text{id} \otimes \Delta)\Delta((C, \Lambda_C); \vec{s}). \end{aligned}$$

This proves the coassociativity.

We also prove the counit property $(\varepsilon \otimes \text{id}) \Delta = \beta_\ell$ by induction on $|\vec{s}|$ with the initial case $|\vec{s}| = 0$ given by the counit property on $\mathbb{Q}\mathfrak{C}$. Suppose that the property is proved for lattice cones with $|\vec{s}| = \ell \geq 0$. Then for $((C, \Lambda_C); \vec{s}) \in \mathfrak{D}\mathfrak{C}$ with $|\vec{s}| = \ell + 1$, there is some $1 \leq i \leq k$ such that $s_i \leq -1$. Then

$$\begin{aligned}
 (\varepsilon \otimes \text{id}) \Delta(C; \vec{s}) &= (\varepsilon \otimes \text{id})(\delta_i \otimes \text{id} + \text{id} \otimes \delta_i) \Delta(C; \vec{s} + e_i) \\
 &= (\varepsilon \delta_i \otimes \text{id} + \varepsilon \otimes \delta_i) \Delta(C; \vec{s} + e_i) \\
 &= (\varepsilon \otimes \delta_i) \Delta(C; \vec{s} + e_i) \\
 &= (\text{id} \otimes \delta_i)(\varepsilon \otimes \text{id}) \Delta(C; \vec{s} + e_i) \\
 &= (\text{id} \otimes \delta_i) \beta_\ell(C; \vec{s} + e_i) \\
 &= \beta_\ell \delta_i(C; \vec{s} + e_i) \\
 &= \beta_\ell(C; \vec{s}).
 \end{aligned}$$

This completes the induction. The proof of $(\text{id} \otimes \varepsilon) \Delta = \beta_r$ is similar.

From the fact that $\mathbb{Q}\mathfrak{D}\mathfrak{C}$ is cograded with the grading in Eq. (18), we have

$$\mathbb{Q}\mathfrak{D}\mathfrak{C} = \mathbb{Q}u(1) \oplus \ker \varepsilon$$

and $\mathbb{Q}\mathfrak{D}\mathfrak{C}^{(0)} = \{(\{\{0\}, \{0\}\}; (0))\}$. Hence $\mathbb{Q}\mathfrak{D}\mathfrak{C}$ is connected. \square

Corollary 3.6. *Let $\mathfrak{C}\mathfrak{h}$ be the set of lattice Chen cones, their faces and their transverse lattice cones in $(\mathbb{R}^\infty, \mathbb{Z}^\infty)$ and $\mathfrak{D}\mathfrak{C}\mathfrak{h} = \mathfrak{C}\mathfrak{h} \times \mathbb{Z}_{\leq 0}^\infty$, then $\mathbb{Q}\mathfrak{C}\mathfrak{h}$ and $\mathbb{Q}\mathfrak{D}\mathfrak{C}\mathfrak{h}$ are sub-coalgebras of $\mathbb{Q}\mathfrak{D}\mathfrak{C}$.*

4. RENORMALISATION ON CHEN CONES

We want to renormalise multiple zeta values, so we consider the space $\mathbb{Q}\mathfrak{D}\mathfrak{C}\mathfrak{h}$. For a lattice cone (C, Λ_C) , one way to regularise the sum

$$\sum_{\vec{n} \in C^o \cap \Lambda_C} 1$$

is to introduce a linear form α on V_k and a parameter ε , and then define

$$\phi(C, \Lambda_C) := \sum_{\vec{n} \in C^o \cap \Lambda_C} e^{\alpha(\vec{n})\varepsilon}.$$

Usually, we assume that α is rational, that is $\alpha(\vec{n}) \in \mathbb{Q}$ for $\vec{n} \in \Lambda_C$.

A problem arises with this regularisation, namely in order for $S(C, \Lambda_C)(\varepsilon)$ to be a Laurent series in ε , we need $\text{Ker}(\alpha) \cap C^o \cap \Lambda_C = \{0\}$ for otherwise there are infinite many 1's in the summation.

Remark 4.1. (a) For a single lattice cone, it is easy to find such a linear function α , but problems can arise to find a linear function well suited for a family of lattice cones. For the family \mathfrak{C} , it is impossible to find a universal α ; take any $v \in \ker(\alpha)$, then α vanishes on $\langle v \rangle$.

(b) For the family of cones in the the first orthant, it is also impossible to find a universal α . This can be reduced to the two dimensional case. Any rational vector v in the open upper half plane defines a cone $\langle v \rangle$ in the first quadrant or a transverse cone $\langle v \rangle = t(C, f)$ of a face f of a two dimensional cone C in the first quadrant. Choosing v in $\text{Ker}(\alpha)$, implies that α vanishes on $\langle v \rangle$. This extends to the closed upper half-plane since $\langle e_1 \rangle$ is a cone in the first quadrant.

However, it is possible to find such an α for a small enough family, for example the family $\mathfrak{C}\mathfrak{h}$.

Proposition 4.2. *A linear form $\alpha = \sum a_i e_i^*$ is negative on all cones in \mathfrak{Ch} if and only if $a_i < a_{i+1} < 0$ for $i \in \mathbb{N}$.*

Proof. In order to give the proof, we first determine the form of the transverse cones to faces of a Chen cone $C := \langle v_1, \dots, v_k \rangle$, where we have set $v_i := e_1 + \dots + e_i$ for $i \geq 1$. For positive integers $p < q$, denote $[p, q] := [p, p+1, \dots, q]$, and $v_{[p,q]} = v_p, v_{p+1}, \dots, v_q$. Then a face of C is of the form

$$F = \langle v_{[j_0, i_1]}, v_{[j_1, i_2]}, \dots, v_{[j_n, i_{n+1}]} \rangle, \quad 0 =: i_0 \leq j_0 \leq i_1 \leq j_1 \leq i_2 \leq j_2 \leq \dots \leq i_n \leq j_n \leq i_{n+1} \leq j_{n+1} := k+1.$$

Here $p \leq q$ means $p+2 \leq q$. Then the transverse cone is generated by $\pi_{F^\perp}(v_m)$ with $i_\ell < m < j_\ell$, for $0 \leq \ell \leq n+1$ with $i_\ell \leq j_\ell$.

First let us compute $\pi_{F^\perp}(e_m)$ for $i_\ell < m < j_\ell$, for $0 \leq \ell \leq n+1$ with $i_\ell \leq j_\ell$. We know that

(a1) if $\ell = 0$, $i_0 \leq j_0$, then

$$e_m = \frac{j_0 - i_0 - 1}{j_0 - i_0} (e_m - e_{j_0}) - \frac{1}{j_0 - i_0} \sum_{i_0 < t < j_0, t \neq m} (e_t - e_{j_0}) + \frac{1}{j_0 - i_0} (v_{j_0}).$$

(a2) if $0 < \ell < n+1$, $i_\ell \leq j_\ell$, then

$$e_m = \frac{j_\ell - i_\ell - 1}{j_\ell - i_\ell} (e_m - e_{j_\ell}) - \frac{1}{j_\ell - i_\ell} \sum_{i_\ell < t < j_\ell, t \neq m} (e_t - e_{j_\ell}) + \frac{1}{j_\ell - i_\ell} (v_{j_\ell}) - \frac{1}{j_\ell - i_\ell} (v_{i_\ell}).$$

(b) if $\ell = n+1$, $i_{n+1} \leq j_{n+1}$, then

$$e_m = e_m.$$

For $0 \leq \ell < n+1$ and $i_\ell < t < j_\ell$, there is $(e_t - e_{j_\ell}) \perp \text{lin}(F)$. For $\ell = n+1$ and $i_{n+1} < t < j_{n+1}$, there is $e_t \perp \text{lin}(F)$. Thus for the projection of e_m we have

(a) if $0 \leq \ell < n+1$, $i_\ell \leq j_\ell$, $i_\ell < m < j_\ell$, then

$$\pi_{F^\perp}(e_m) = \frac{j_\ell - i_\ell - 1}{j_\ell - i_\ell} (e_m - e_{j_\ell}) - \frac{1}{j_\ell - i_\ell} \sum_{i_\ell < t < j_\ell, t \neq m} (e_t - e_{j_\ell}).$$

(b) if $\ell = n+1$, $i_{n+1} \leq j_{n+1}$, $i_{n+1} < m < j_{n+1}$, then

$$\pi_{F^\perp}(e_m) = e_m.$$

Therefore,

(a) if $0 \leq \ell < n+1$, $i_\ell \leq j_\ell$, $i_\ell < m < j_\ell$, then

$$\begin{aligned} \pi_{F^\perp}(v_m) &= \frac{j_\ell - m}{j_\ell - i_\ell} \sum_{i_\ell < t \leq m} (e_t - e_{j_\ell}) - \frac{m - i_\ell}{j_\ell - i_\ell} \sum_{m < t < j_\ell} (e_t - e_{j_\ell}) \\ &= \frac{j_\ell - m}{j_\ell - i_\ell} \sum_{i_\ell < t \leq m} e_t - \frac{m - i_\ell}{j_\ell - i_\ell} \sum_{m < t < j_\ell} e_t. \end{aligned}$$

(b) if $\ell = n+1$, $i_{n+1} \leq j_{n+1}$, $i_{n+1} < m < j_{n+1}$, then $\pi_{F^\perp}(v_m) = e_{i_{n+1}+1} + \dots + e_m$.

We are now ready to prove the proposition, noting that α is negative on a transverse cone if and only if it is so on its generators $\pi_{F^\perp}(v_m)$, $i_\ell < m < j_\ell$, $0 \leq \ell \leq n+1$.

Let α be negative on all transverse cones to faces the cone $C = \langle v_1, \dots, v_k \rangle$, $k \geq 1$. Then the transverse cone for the face $\langle v_1, \dots, \hat{v}_i, \dots, v_k \rangle$ (the cone spanned by v_1, \dots, v_k except v_i), $i = 1, \dots, k-1$, is spanned by $\frac{1}{2}(e_i - e_{i+1})$, by the above Case (a). Then applying α to this transverse cone, we have $a_i < a_{i+1}$. Now for the cone $\langle v_1, \dots, v_{k-1} \rangle$, by Case (b), the transverse cone is generated by e_k , applying α yields $a_k < 0$. This is what we need.

Conversely, suppose that $\alpha = \sum a_i e_i^*$ satisfies $a_i < a_{i+1} < 0$. Clearly, α is negative on C and its faces. It is also negative on $\pi_{F^\perp}(v_m)$ in the Case (b). For $\pi_{F^\perp}(v_m)$ in Case (a), using the fact

$$\frac{j_\ell - m}{j_\ell - i_\ell} \sum_{i_\ell < t \leq m} 1 = \frac{m - i_\ell}{j_\ell - i_\ell} \sum_{m < t \leq j_\ell} 1,$$

we find $\alpha(\pi_{F^\perp}(v_m)) < 0$. Therefore α is negative on all transverse cones. \square

We now fix a linear function $\alpha = \sum a_i e_i^*$ with $a_i < a_{i+1} < 0$, and for $((C, \Lambda_C), \vec{s}) \in \mathfrak{DC}\mathfrak{h}$, we set

$$(19) \quad \phi((C, \Lambda_C), \vec{s}) = \sum_{\vec{n} \in \Lambda_C \cap C^o} \frac{e^{\alpha(\vec{n})\varepsilon}}{\vec{n}^{\vec{s}}},$$

Applying the same proof as for Lemma 4.4 in [4], we have

Lemma 4.3. *The map $\phi(C, \Lambda_C)$ is a meromorphic function in ε for any coloured lattice cone $((C, \Lambda_C), \vec{s})$ in $\mathfrak{DC}\mathfrak{h}$.*

This gives rise to a linear map:

$$\phi : \mathbb{Q}\mathfrak{DC}\mathfrak{h} \rightarrow \mathbb{C}[\varepsilon^{-1}, \varepsilon]$$

to which we can then apply Connes-Kreimer's renormalisation scheme on the coalgebra of Chen cones as in Theorem 2.5, without bothering about the product structure. So, applying the induction formula with $(R, P) = (\mathbb{C}[\varepsilon^{-1}, \varepsilon], -\pi_+)$, where π_+ is the projection to the holomorphic part, we have

$$\phi = \phi_-^{*(-1)} * \phi_+,$$

where $\phi_-^{*(-1)}$ is the holomorphic part and ϕ_+ is the polar part. Here ϕ_- takes values in $\mathbb{C}[[\varepsilon]]$ and ϕ_+ takes values in $\mathbb{C}[\varepsilon^{-1}]$.

Let us define renormalised multiple zeta values as

$$(20) \quad \zeta^{\text{ren}}((C, \Lambda_C), \vec{s}) := \phi_-^{*(-1)}((C, \Lambda_C), \vec{s})(0).$$

We will see that the renormalised multiple zeta values do not depend on the parameters a_i , a fact which might seem surprising at first glance and that will be proved in the sequel. An important consequence is that the parameters can be seen as formal parameters, thus allowing for a regularisation in a more general situation than the one of Chen cones considered here.

5. RENORMALISED CONICAL ZETA VALUES

As we previously discussed, it is impossible to find a universal linear function α which would regularise all cones simultaneously, but it is possible to find one for the family of Chen cones; in the Chen cone case, we renormalise along a direction $\vec{d} := (a_1, a_2, \dots)\varepsilon$. Since the parameter ε can be viewed as a re-scaling of variables, this suggests to replace the parameters $\vec{d} := (a_1, a_2, \dots, a_k)$ by the variables $\vec{\varepsilon} = \sum \varepsilon_i e_i^* \in V^*$, where $\varepsilon_1 := a_1\varepsilon, \varepsilon_2 := a_2\varepsilon, \dots, \varepsilon_k := a_k\varepsilon$, and to define

$$(21) \quad S_k^o((C, \Lambda_C); \vec{s})(\vec{\varepsilon}) := \sum_{\vec{n} \in \Lambda_C \cap C^o} \frac{e^{\langle \vec{n}, \vec{\varepsilon} \rangle}}{\vec{n}^{\vec{s}}} = \sum_{(n_1, \dots, n_k) \in C^o \cap \Lambda_C} \frac{e^{n_1 \varepsilon_1} \dots e^{n_k \varepsilon_k}}{n_1^{s_1} \dots n_k^{s_k}} = \sum_{\vec{n} \in C^o \cap \Lambda_C} \frac{e^{\langle \vec{n}, \vec{\varepsilon} \rangle}}{\vec{n}^{\vec{s}}}$$

for a simplicial lattice cone (so in particular it is strongly convex) $(C, \Lambda_C) \in \mathfrak{C}$ with $C \subset \mathbb{R}^k$ and where we have set $\vec{n}^{\vec{s}} = n_1^{s_1} \cdots n_k^{s_k}$ with $\vec{n} := (n_1, \dots, n_k) \in \Lambda_C$ and $\vec{s} = (s_1, \dots, s_k) \in \mathbb{Z}_{\leq 0}^k$. The sum (21) is absolutely convergent on

$$\check{C}^- := \left\{ \vec{\varepsilon} := \sum_{i=1}^k \varepsilon_i e_i^* \mid \langle \vec{x}, \vec{\varepsilon} \rangle < 0 \text{ for all } \vec{x} \in C \right\},$$

which like C , has dimension k .

Remark 5.1. With our convention that $0^s = 1$ for s with $\text{Re}(s) \leq 0$, the function $S_k^o((C, \Lambda_C); \vec{s})(\vec{\varepsilon})$ in the variables $\vec{\varepsilon} = \sum \varepsilon_i e_i^*$ does not depend on the choice of $k \geq 1$ such that $C \subseteq V_k$ and $\vec{s} \in \mathbb{Z}_{\leq 0}^k$. Thus we will suppress the subscript k in the sum.

Choosing the above multivariate regularisation implies that—in contrast to Connes and Kreimer’s renormalisation scheme—the range space is no longer the space of Laurent series. The new target space is a space of multivariate meromorphic germs discussed in [3] which is not a Rota-Baxter algebra, thus requiring² the generalised version of Connes and Kreimer’s renormalisation scheme corresponding to Theorem 2.5.

5.1. Regularisations. The function $S^o((C, \Lambda_C), \vec{s})$ is a very specific type of meromorphic function, for it has linear poles. We briefly review the relevant definitions, and refer the reader to [3] for a more detailed discussion.

Definition 5.2. Let k be a positive integer.

- (a) A **germ of meromorphic functions at 0** on \mathbb{C}^k is the quotient of two holomorphic functions in a neighborhood of 0 inside \mathbb{C}^k .
- (b) A germ of meromorphic functions $f(\vec{\varepsilon})$ on \mathbb{C}^k is said to have **linear poles at zero with rational coefficients** if there exist vectors $L_1, \dots, L_n \in \Lambda_k \otimes \mathbb{Q}$ (possibly with repetitions) such that $f \prod_{i=1}^n L_i$ is a holomorphic germ at zero whose Taylor expansion has rational coefficients.
- (c) We will denote by $\mathcal{M}_{\mathbb{Q}}(\mathbb{C}^k)$ the set of germs of meromorphic functions on \mathbb{C}^k with linear poles at zero with rational coefficients. It is a linear subspace over \mathbb{Q} .

Composing with the projection $\mathbb{C}^{k+1} \rightarrow \mathbb{C}^k$ dual to the inclusion $j_k : \mathbb{C}^k \rightarrow \mathbb{C}^{k+1}$ then yields the embedding

$$\mathcal{M}_{\mathbb{Q}}(\mathbb{C}^k) \hookrightarrow \mathcal{M}_{\mathbb{Q}}(\mathbb{C}^{k+1}),$$

thus giving rise to the direct limit

$$\mathcal{M}_{\mathbb{Q}}(\mathbb{C}^{\infty}) := \varinjlim \mathcal{M}_{\mathbb{Q}}(\mathbb{C}^k) = \bigcup_{k=1}^{\infty} \mathcal{M}_{\mathbb{Q}}(\mathbb{C}^k).$$

Proposition 5.3. [3] *There is a direct sum decomposition*

$$\mathcal{M}_{\mathbb{Q}}(\mathbb{C}^{\infty}) = \mathcal{M}_{\mathbb{Q},-}(\mathbb{C}^{\infty}) \oplus \mathcal{M}_{\mathbb{Q},+}(\mathbb{C}^{\infty}).$$

Thus we have the projection map

$$(22) \quad \pi_+ : \mathcal{M}_{\mathbb{Q}}(\mathbb{C}^{\infty}) \rightarrow \mathcal{M}_{\mathbb{Q},+}(\mathbb{C}^{\infty}).$$

²As observed in [4], the renormalised conical values we derive here by means of a multivariate Algebraic Birkhoff Factorisation, can alternatively be derived directly from the derivatives of the exponential sums on cones by means of the projection onto the holomorphic part of the meromorphic germs they give rise to, an alternative renormalisation method which gives rise to the same conical values.

A subdivision technique then yields the following.

Proposition-Definition 5.1. [4] *For any simplicial lattice cone (C, Λ_C) , the map $S^o((C, \Lambda_C); \vec{s})(\vec{\varepsilon})$ defines an element in $\mathcal{M}_{\mathbb{Q}}(\mathbb{C}^\infty)$.*

For a general lattice cone (C, Λ_C) , the germ of functions $\sum_{F \in \mathcal{F}^o(\underline{C})} S^o((F, \Lambda_F); \vec{s})$ does not depend on the choice of the simplicial subdivision $\underline{C} = \{(C_i, \Lambda_{C_i})\}_{i \in [n]}$ of (C, Λ_C) . Thus we extend (21) to any lattice cone setting

$$S^o((C, \Lambda_C); \vec{s}) := \sum_{F \in \mathcal{F}^o(\underline{C})} S^o((F, \Lambda_F); \vec{s}),$$

for any simplicial subdivision $\underline{C} = \{(C_i, \Lambda_{C_i})\}_{i \in [n]}$ of (C, Λ_C) .

Consequently, we have a linear map

$$S^o : \mathbb{Q}\mathfrak{DC} \rightarrow \mathcal{M}_{\mathbb{Q}}(\mathbb{C}^\infty), \quad ((C, \Lambda_C); \vec{s}) \mapsto S^o((C, \Lambda_C); \vec{s}).$$

By definition, the following conclusion holds.

Corollary 5.4. *Let (C, Λ_C) be a lattice cone and let $\underline{C} = \{(C_1, \Lambda_{C_1}), \dots, (C_r, \Lambda_{C_r})\}$ be a subdivision of C . Then for $\vec{s} \in \mathbb{Z}_{\leq 0}^k$ we have*

$$S^o((C, \Lambda_C); \vec{s}) = \sum_{F \in \mathcal{F}^o(\underline{C})} S^o((F, \Lambda_C \cap \text{lin}(F)); \vec{s})$$

in $\mathcal{M}_{\mathbb{Q}}(\mathbb{C}^\infty)$.

One advantage to work with this multivariate regularisation is that the target space is stable under partial derivatives, and we thus have a linear map compatible with coderivatives... Let

$$\partial_i = \frac{\partial}{\partial \varepsilon_i}.$$

By an analytic continuation argument, we have the following relations between regularised conical zeta values.

Proposition 5.5. *For the linear map*

$$S^o : \mathbb{Q}\mathfrak{DC} \rightarrow \mathcal{M}_{\mathbb{Q}}(\mathbb{C}^\infty)$$

and any $i \in \mathbb{Z}_{>0}$,

$$S^o \delta_i = \partial_i S^o.$$

That means for any $((C, \Lambda_C), \vec{s})$ in \mathfrak{DC} , we have

$$S^o((C, \Lambda_C); \vec{s})(\vec{\varepsilon}) = \partial^{-\vec{s}} S^o(C, \Lambda_C)(\vec{\varepsilon}),$$

where $\partial^{-\vec{s}} = \partial_1^{-s_1} \dots \partial_k^{-s_k}$.

Proof. For a given $\vec{s} \in \mathbb{Z}_{\leq 0}^k$ and a simplicial lattice cone $(C, \Lambda_C) \in \mathfrak{C}$ with $C \subset \mathbb{R}^k$, by absolute convergence we have

$$\partial_i S^o((C, \Lambda_C); \vec{s})(\vec{\varepsilon}) = S^o((C, \Lambda_C); \vec{s} - e_i)(\vec{\varepsilon}) = S^o(\delta_i((C, \Lambda_C); \vec{s}))(\vec{\varepsilon})$$

for $\vec{\varepsilon} \in \check{C}^-$. Therefore by analytic continuation, in $\mathcal{M}_{\mathbb{Q}}(\mathbb{C}^\infty)$, we have

$$\partial_i S^o((C, \Lambda_C); \vec{s})(\vec{\varepsilon}) = S^o(\delta_i((C, \Lambda_C); \vec{s}))(\vec{\varepsilon}),$$

that is,

$$S^o \delta_i = \partial_i S^o$$

for any simplicial lattice cone. Then by definition of S^o , $S^o \delta_i = \partial_i S^o$ holds in general. \square

5.2. Renormalisation. We now equip \mathbb{R}^∞ with an inner products $Q(\cdot, \cdot)$. This allows us to construct the coalgebra $\mathbb{Q}\mathfrak{D}\mathfrak{C}$ from transverse lattice cones introduced in Section 2, and to apply [3, Theorem 4.2] in view of the linear decomposition

$$\mathcal{M}_{\mathbb{Q}}(\mathbb{C}^\infty) = \mathcal{M}_{\mathbb{Q},+}(\mathbb{C}^\infty) \oplus \mathcal{M}_{\mathbb{Q},-}(\mathbb{C}^\infty).$$

Since $\mathcal{M}_{\mathbb{Q},+}(\mathbb{C}^\infty)$ is a unitary subalgebra, the Algebraic Birkhoff Factorisation in Theorem 2.5 applies, with $\mathbf{C} = \mathbb{Q}\mathfrak{D}\mathfrak{C}$ and

$$A = \mathcal{M}_{\mathbb{Q}}(\mathbb{C}^\infty), \quad A_1 = \mathcal{M}_{\mathbb{Q},+}(\mathbb{C}^\infty), \quad A_2 = \mathcal{M}_{\mathbb{Q},-}(\mathbb{C}^\infty), \quad P = \pi_+ : \mathcal{M}_{\mathbb{Q}}(\mathbb{C}^\infty) \rightarrow \mathcal{M}_{\mathbb{Q},+}(\mathbb{C}^\infty).$$

We consequently obtain the following theorem.

Theorem 5.6. (Algebraic Birkhoff Factorisation for conical zeta values) *For the linear map*

$$S^o : \mathbb{Q}\mathfrak{D}\mathfrak{C} \rightarrow \mathcal{M}_{\mathbb{Q}}(\mathbb{C}^\infty),$$

there exist unique linear maps $S_1^o : \mathbb{Q}\mathfrak{D}\mathfrak{C} \rightarrow \mathcal{M}_{\mathbb{Q},+}(\mathbb{C}^\infty)$ and $S_2^o : \mathbb{Q}\mathfrak{D}\mathfrak{C} \rightarrow \mathbb{Q} + \mathcal{M}_{\mathbb{Q},-}(\mathbb{C}^\infty)$, with $S_1^o(\{0\}, \{0\}) = 1$, $S_2^o(\{0\}, \{0\}) = 1$, such that

$$(23) \quad S^o = (S_1^o)^{*(-1)} * S_2^o.$$

The same theorem applies to the sub-coalgebra $\mathbb{Q}\mathfrak{C}$, which yields a factorisation of $S^o : \mathbb{Q}\mathfrak{C} \rightarrow \mathcal{M}_{\mathbb{Q}}(\mathbb{C}^\infty)$, giving rise to two linear maps $S_1^o : \mathbb{Q}\mathfrak{C} \rightarrow \mathcal{M}_{\mathbb{Q},+}(\mathbb{C}^\infty)$ and $S_2^o : \mathbb{Q}\mathfrak{C} \rightarrow \mathbb{Q} + \mathcal{M}_{\mathbb{Q},-}(\mathbb{C}^\infty)$. We can legitimately use the same notation as in Theorem 5.6 since they correspond to the restriction of the linear maps in Theorem 5.6 as a result of the uniqueness of the factorisation.

In [4], we identify S_2^o with the exponential integral and give a formula for

$$\mu^o(C, \Lambda_C) := (S_1^o)^{*(-1)}(C, \Lambda_C)$$

as follows.

Proposition 5.7. *As a linear map on $\mathbb{Q}\mathfrak{C}$, we have*

$$S_2^o = I,$$

$$\mu^o = \pi_+ S^o.$$

Here I is the exponential integral on lattice cones [4] defined as follows on simplicial cones and then extended to any cone by the subdivision property. If $v_1, \dots, v_k \in \Lambda_C$ is a set of primary generators of a simplicial cone C , and u_1, \dots, u_k a basis of Λ_C , for $1 \leq i \leq k$, let $v_i = \sum_{j=1}^k a_{ji} u_j$, $a_{ji} \in$

\mathbb{Z} . Define linear functions $L_i := L_{v_i} := \sum_{j=1}^k a_{ji} \langle u_j, \vec{\epsilon} \rangle$ and let $w(C, \Lambda_C)$ denote the absolute value of the determinant of the matrix $[a_{ij}]$, then

$$(24) \quad I(C, \Lambda_C)(\vec{\epsilon}) := (-1)^k \frac{w(C, \Lambda_C)}{L_1 \cdots L_k}.$$

In general we also have

Proposition 5.8. For $((C, \Lambda_C); \vec{s}) \in \mathbb{Q}\mathfrak{DC}$, we have

$$(25) \quad S_1^o((C, \Lambda_C); \vec{s}) = \partial^{-\vec{s}} S_1^o(C, \Lambda_C), \quad S_2^o((C, \Lambda_C); \vec{s}) = \partial^{-\vec{s}} S_2^o(C, \Lambda_C)$$

and

$$(26) \quad \mu^o = \pi_+ S^o.$$

Proof. By Proposition 5.5, S^o are compatible with the coderivations on $\mathbb{Q}\mathfrak{DC}$ and derivations on $\mathcal{M}_{\mathbb{Q}}(\mathbb{C}^\infty)$. The conclusion then follows from Theorem 2.5. \square

For $((C, \Lambda_C); \vec{s}) \in \mathfrak{DC}$ the expressions $\mu^o((C, \Lambda_C); \vec{s}) = (S_1^o)^{*(-1)}((C, \Lambda_C); \vec{s})$ in the Algebraic Birkhoff Factorisation of S^o is a germ of holomorphic functions which we can therefore evaluate at 0.

Definition 5.9. The value

$$\zeta^o((C, \Lambda_C); \vec{s}) := (S_1^o)^{*(-1)}((C, \Lambda_C); \vec{s})(0)$$

is called the **renormalised open conical zeta value** of $((C, \Lambda_C); \vec{s})$.

In particular, this definition applies to cones in \mathfrak{Ch} and \mathfrak{DCh} .

Corollary 5.10. The germs of functions $(S_1^o)^{*(-1)}(C, \Lambda_C)$ are generating functions of renormalised open conical zeta values at nonpositive integers. More precisely, for a lattice cone $(C, \Lambda_C) \in \mathfrak{C}$, we have

$$(27) \quad (S_1^o)^{*(-1)}(C, \Lambda_C)(\vec{\varepsilon}) = \sum_{\vec{r} \in \mathbb{Z}_{\geq 0}^k} \zeta^o((C, \Lambda_C); -\vec{r}) \frac{\vec{\varepsilon}^{\vec{r}}}{\vec{r}!}.$$

Proof. By Eq. (25), we have

$$\partial_{\vec{\varepsilon}}^{\vec{r}} (S_1^o)^{*(-1)}(C, \Lambda_C)(0) = (S_1^o)^{*(-1)}((C, \Lambda_C); -\vec{r})(0) = \zeta^o((C, \Lambda_C); -\vec{r}),$$

as needed. \square

6. COMPARISON OF THE TWO RENORMALISATION SCHEMES

So far, we have two approaches to renormalise sums on Chen cones, which can be related by means of a restriction $\vec{\varepsilon} = \vec{a} \varepsilon$ along a direction \vec{a} : the first one by which the Algebraic Birkhoff Factorisation procedure is implemented after restricting, the second one by which the Algebraic Birkhoff Factorisation procedure is implemented before restricting.

Under the restriction along a direction \vec{a} , the splittings of the target space in the two approaches differ as it can be seen on the following counterexample which shows that evaluation $\mathcal{E}_{\vec{a}}$ along a given direction $\vec{a} \varepsilon$ does not commute with the projection π_+ :

$$\pi_+ \circ \mathcal{E}_{\vec{a}} \neq \mathcal{E}_{\vec{a}} \circ \pi_+,$$

where the projection π_+ on the left hand side is the one on $\mathcal{M}_{\mathbb{Q}}(\mathbb{C}^\infty)$ and the one on the right hand side is on $\mathcal{M}_{\mathbb{Q}}(\mathbb{C})$.

Counterexample 6.1. Let $f(\varepsilon_1, \varepsilon_2) := \frac{\varepsilon_1}{\varepsilon_2}$, then

$$\pi_+ \circ \mathcal{E}_{\vec{a}}(f) = \frac{a_1}{a_2} \neq 0 = \mathcal{E}_{\vec{a}} \circ \pi_+(f).$$

But surprisingly, these two renormalisation procedures give the same renormalised values for Chen cones.

Proposition 6.2. *For Chen cones, the factorisations obtained by*

- *first implementing the Algebraic Birkhoff Factorisation on the exponential sum S^o and then restricting along a direction $\vec{d}\varepsilon$, and*
- *first restricting the exponential sum S^o along a direction $\vec{d}\varepsilon$ and then implementing the Algebraic Birkhoff Factorisation*

coincide.

Proof. We first investigate the first renormalisation procedure. Since the Algebraic Birkhoff Factorisation applied to the exponential sum S^o on cones boils down to the Euler-Maclaurin formula on cones [4], we have that on \mathbb{QC}

$$(28) \quad S^o = \mu^o * I,$$

where $*$ is the convolution associated with the coproduct on lattice cones. For any lattice cone (C, Λ_C) , $\mu^o(C, \Lambda_C)$ is holomorphic and $I(C, \Lambda_C)$ is a sum of simple fractions. By Proposition 5.8, differentiating yields for any lattice cone (C, Λ_C) and any \vec{s} , a holomorphic function $\mu^o((C, \Lambda_C); \vec{s})$ and a sum $I((C, \Lambda_C); \vec{s})$ of simplicial fractions. Now, restricting along the direction $\vec{\varepsilon} = \vec{d}\varepsilon$ yields for any lattice cone (C, Λ_C) and \vec{s} , a map $\mu^o((C, \Lambda_C); \vec{s})|_{\vec{\varepsilon}=\vec{d}\varepsilon}$ in $\mathbb{Q}[[\varepsilon]]$. Furthermore, the restriction $I((C, \Lambda_C); \vec{s})|_{\vec{\varepsilon}=\vec{d}\varepsilon}$ lies in $\mathbb{Q}[\varepsilon^{-1}]\varepsilon^{-1}$ if $((C, \Lambda_C); \vec{s}) \neq ((\{0\}, \{0\}), \vec{0})$ as a sum of restricted simplicial fractions. So if we let

$$\tilde{\mu}((C, \Lambda_C); \vec{s})(\varepsilon) = \mu^o((C, \Lambda_C); \vec{s})(\vec{\varepsilon})|_{\vec{\varepsilon}=\vec{d}\varepsilon},$$

and

$$\tilde{I}((C, \Lambda_C); \vec{s})(\varepsilon) = I((C, \Lambda_C); \vec{s})(\vec{\varepsilon})|_{\vec{\varepsilon}=\vec{d}\varepsilon},$$

with $\phi((C, \Lambda_C); \vec{s})(\varepsilon) = S^o((C, \Lambda_C); \vec{s})(\vec{\varepsilon})|_{\vec{\varepsilon}=\vec{d}\varepsilon}$ as in (19), we have

$$\phi = \tilde{\mu} * \tilde{I},$$

where $\tilde{\mu}((C, \Lambda_C); \vec{s}) \in \mathbb{Q}[[\varepsilon]]$ and $\tilde{I}((C, \Lambda_C); \vec{s}) \in \mathbb{Q} + \mathbb{Q}[\varepsilon^{-1}]\varepsilon^{-1}$.

The alternative renormalisation procedure is to implement Algebraic Birkhoff Factorisation on the restricted map ϕ , which yields a factorisation

$$\phi = \phi_-^{*(-1)} * \phi_+,$$

with $\phi_-^{*(-1)}((C, \Lambda_C); \vec{s}) \in \mathbb{C}[[\varepsilon]]$, and $\phi_+((C, \Lambda_C); \vec{s}) \in \mathbb{C}[\varepsilon^{-1}]$.

Thus both factorisations are for linear maps between the same spaces. Now the standard argument of the uniqueness of the Algebraic Birkhoff Factorisation then shows that the two factorisations coincide. \square

Corollary 6.3. *The renormalised multiple zeta values do not depend on the parameters a_1, a_2, \dots .*

Let us illustrate the two approaches on a simple example. To simplify notations, for k linear forms L_1, \dots, L_k , we set

$$(29) \quad [L_1, \dots, L_k] := \frac{e^{L_1}}{1 - e^{L_1}} \frac{e^{L_1+L_2}}{1 - e^{L_1+L_2}} \cdots \frac{e^{L_1+L_2+\dots+L_k}}{1 - e^{L_1+L_2+\dots+L_k}}.$$

and

$$(30) \quad \frac{e^\varepsilon}{1 - e^\varepsilon} = -\frac{1}{\varepsilon} + h(\varepsilon).$$

Example 6.4. For $k = 2$ and the Chen cone $\langle e_1, e_1 + e_2 \rangle$, we have

$$S^o(\langle e_1, e_1 + e_2 \rangle, \Lambda_2) = [\varepsilon_1, \varepsilon_2],$$

$$\begin{aligned} \pi_+([\varepsilon_1, \varepsilon_2]) &= \pi_+ \left(\left(-\frac{1}{\varepsilon_1} + h(\varepsilon_1) \right) \left(-\frac{1}{\varepsilon_1 + \varepsilon_2} + h(\varepsilon_1 + \varepsilon_2) \right) \right) \\ &= \pi_+ \left(-\frac{h(\varepsilon_1 + \varepsilon_2)}{\varepsilon_1} - \frac{h(\varepsilon_1)}{\varepsilon_1 + \varepsilon_2} + h(\varepsilon_1)h(\varepsilon_1 + \varepsilon_2) \right) \\ &= -\frac{h(\varepsilon_1 + \varepsilon_2) - h(\varepsilon_2)}{\varepsilon_1} - \frac{h(\varepsilon_1) - h\left(\frac{\varepsilon_1 - \varepsilon_2}{2}\right)}{\varepsilon_1 + \varepsilon_2} + h(\varepsilon_1)h(\varepsilon_1 + \varepsilon_2). \end{aligned}$$

So

$$\pi_+([\varepsilon_1, \varepsilon_2])|_{(a_1\varepsilon, a_2\varepsilon)} = -\frac{h((a_1 + a_2)\varepsilon) - h(a_2\varepsilon)}{a_1\varepsilon} - \frac{h(a_1\varepsilon) - h\left(\frac{(a_1 - a_2)\varepsilon}{2}\right)}{(a_1 + a_2)\varepsilon} + h(a_1\varepsilon)h((a_1 + a_2)\varepsilon).$$

Evaluating at $\varepsilon = 0$ yields

$$\zeta(0, 0) = -\frac{(a_1 + a_2) - a_2}{a_1} h'(0) - \frac{\frac{a_1 + a_2}{2}}{a_1 + a_2} h'(0) + h(0)^2 = -\frac{3}{2} h'(0) + h(0)^2 = \frac{3}{8}.$$

On the other hand, to use formula (20) to find $\phi_-^{*(-1)}$ needs more involved computations. We easily get

$$\phi_-^{*(-1)}(\langle e_1 \rangle, \mathbb{Z}e_1) = h(a_1\varepsilon),$$

and

$$\phi_-^{*(-1)}(\langle e_1 + e_2 \rangle, \mathbb{Z}(e_1 + e_2)) = h((a_1 + a_2)\varepsilon).$$

The reduced coproduct applied to the two dimension Chen cone reads

$$\Delta'(\langle e_1, e_1 + e_2 \rangle, \Lambda_2) = (\langle e_2 \rangle, \mathbb{Z}e_2) \otimes (\langle e_1 \rangle, \mathbb{Z}e_1) + (\langle e_1 - e_2 \rangle, \mathbb{Z}\frac{e_1 - e_2}{2}) \otimes (\langle e_1 + e_2 \rangle, \mathbb{Z}(e_1 + e_2)).$$

Thus

$$\begin{aligned} &\phi_- (\langle e_1, e_1 + e_2 \rangle, \Lambda_2) \\ &= -P \left(\left(-\frac{1}{a_1\varepsilon} + h(a_1\varepsilon) \right) \left(-\frac{1}{(a_1 + a_2)\varepsilon} + h((a_1 + a_2)\varepsilon) \right) \right. \\ &\quad \left. + \left(-h(a_2\varepsilon) \right) \left(-\frac{1}{a_1\varepsilon} + h(a_1\varepsilon) \right) + \left(-h((a_1 - a_2)\varepsilon/2) \right) \left(-\frac{1}{(a_1 + a_2)\varepsilon} + h((a_1 + a_2)\varepsilon) \right) \right) \\ &= \frac{h((a_1 + a_2)\varepsilon) - h(a_2\varepsilon)}{a_1\varepsilon} + \frac{h(a_1\varepsilon) - h((a_1 - a_2)\varepsilon/2)}{(a_1 + a_2)\varepsilon} - h(a_1\varepsilon)h((a_1 + a_2)\varepsilon) \\ &\quad + h(a_2\varepsilon)h(a_1\varepsilon) + h((a_1 - a_2)\varepsilon/2)h((a_1 + a_2)\varepsilon). \end{aligned}$$

Now by the equation

$$\begin{aligned} &\phi_- (\langle e_1, e_1 + e_2 \rangle, \Lambda_2) + \phi_-^{*(-1)}(\langle e_1, e_1 + e_2 \rangle, \Lambda_2) \\ &+ \phi_- (\langle e_2 \rangle, \mathbb{Z}e_2) \phi_-^{*(-1)}(\langle e_1 \rangle, \mathbb{Z}e_1) + \phi_- (\langle e_1 - e_2 \rangle, \mathbb{Z}\frac{e_1 - e_2}{2}) \phi_-^{*(-1)}(\langle e_1 + e_2 \rangle, \mathbb{Z}(e_1 + e_2)) \\ &= 0, \end{aligned}$$

we have

$$\phi_-^{*(-1)}(\langle e_1, e_1 + e_2 \rangle, \Lambda_2) = -\frac{h((a_1 + a_2)\varepsilon) - h(a_2\varepsilon)}{a_1\varepsilon} - \frac{h(a_1\varepsilon) - h\left(\frac{(a_1 - a_2)\varepsilon}{2}\right)}{(a_1 + a_2)\varepsilon} + h(a_1\varepsilon)h((a_1 + a_2)\varepsilon).$$

This agrees with $\pi_+([\varepsilon_1, \varepsilon_2])|_{(a_1\varepsilon, a_2\varepsilon)}$.

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